**Unit 14 – Nonlinear Systems**

Goals/Rationale

In previous units students learned about the power and usefulness of classifying equilibrium solutions to understand the structure of the solution space. This was true for both first order DEs and for systems of linear DEs. Similarly, classifying equilibria for nonlinear systems is equally powerful for understanding the structure of the solution space – and that is the goal of this unit. To build intuition the unit starts with the contextually rich situation of a swinging pendulum that they explore graphically, algebraically, and with a GeoGebra applet.

By the end of the unit students will have been introduced to the use of the Jacobian to linearize nonlinear systems. To build intuition for this method, students are first guided through a linear stability analysis for a first order DE. They then explore a nonlinear system by first algebraically finding the equilibrium solutions and then they use a GeoGebra applet to make a tentative classification for each equilibrium. Students are then stepped through a linear stability analysis for this same nonlinear system, which is then formalized with the Jacobian. The unit ends by returning to the pendulum problem to classify equilibria using this new technique.

**Pages 14.1-14.2 – In the swing of things**

Implementation Notes

*Problems 1-2* – In problem 1 students image what graphs in phase plane (angular positive (position) vs angular velocity) would look like for a pendulum that can swing 360 degrees. Before students create their graphs, the instructor could ask them to describe in words what will happen and/or to use their pen or pencil to show what will happen to the pendulum if Debra hits the pendulum with a small amount of force and what will happen if she takes a big swing at the pendulum. **Remind students that the pendulum can rotate in a complete circle.** Students will most likely have a correct image of what will happen, but their graphs are likely to look more like a picture of the situation. The conceptual challenge is that time is implicit in the graph. It is ok for them to produce incorrect graphs, and in fact this can lead to some productive outcomes. Here are some possible questions to ask about students’ graphs.

* What does a negative angle mean in terms of the motion of the pendulum?
* What does a negative velocity mean in terms of the motion of the pendulum?
* How does your graph here compare to the position-velocity graphs of the spring mass studied in Unit 10?

For problem 2, students are likely to either say there is one equilibrium solution (corresponding to the origin). If their graphs depict a pendulum with a least one full revolution then they may realize that there is more than one equilibrium. It is less likely that they will think about the unstable equilibrium. It is NOT necessary for students to have this all figured out at this point as the subsequent problems will address these ideas.

*Problems 3-6* – In problem 3, students are given the second order DE for the pendulum context (derivation of the equation is left to homework problem 5) and they must use this equation to determine the number of equilibrium solutions. If they had not figured out in the previous problem that there are theoretically an infinite number of stable (or spiral sink) and an infinite number of unstable (or spiral source) equilibrium solutions then be sure that this is discussed here and revisit their graphs from problem 1.

*Problem 4* – Students are asked to explain why the sin(Ɵ) is approximately theta for small theta. Drawing a graph of y=sin(Ɵ) and y=Ɵ is one way they might justify this. Another way (one that the instructor can bring in) is to use Taylor series. Students are then to replace sin(theta) with theta to create a linear system and then use the techniques and ideas from the unit on linear systems to determine the type of equilibrium solution at (0,0).

*Problem 5* – Students must realize that they cannot use the linear system to classify the equilibrium solution at pi (push them to explain why) and hence their reasoning at this point will necessarily be grounded in the context. The main point to bring out here is that there is an initial velocity for which the pendulum could theoretically balance precariously upside down. A greater initial velocity would result in the pendulum making a complete revolution.

*Problem 6* – In this problem students use a GeoGebra applet to explore the range of initial velocities that result in one complete revolution. Homework problem 4 follows up on this exploration.

**Pages 14.3 – 14.4 – Linearization and linear stability analysis**

*Problems 7-8* – Problem 7 gives a table that steps students through the process of linearizing and conducting a stability analysis on the linearized system for a first order nonlinear DE. Students will likely need some help realizing that the linearization process shifts the origin so that it is at the equilibrium point (first at (1,0) and then again at (-1,0) for problem 8b). The table can be completed with an instructor led discussion (especially since students will need some guidance on how the linearization makes use of a shifted coordinate system) and then students can repeat the process for problem 8.

*Problem 9* – In this problem students build intuition for the phase plane of a nonlinear system (where *dx/dt* is the same nonlinear DE explored in the previous problem). They algebraically determine the equilibrium solutions (1, -1) and (-1, 1) and use a GeoGebra applet to graphically classify the equilibria.

*Problems 10-11* – these two problems are similar to problems 7 and 8 in that students are first stepped through the linearization process via a table (again, this can be done as a whole class discussion) and then students repeat the method for the second equilibrium solution.

*Problem 12* – formally introduces the Jacobian to consolidate work done in problems 10-11.

*Problem 13* – In this problem students return to the pendulum problem and use what they have learned about the Jacobian to classify the equilibrium solutions.

**Personal Reflections on Unit 14**